1. Let M be a regular surface in R^3 defined by

$$X(u,v) = (u-v,\ v,\ u^2-2uv).$$

(a) $X_u = \begin{pmatrix} 1 \\ 0 \\ 2u-2v \end{pmatrix}$, $X_v = \begin{pmatrix} -1 \\ 1 \\ -2u \end{pmatrix}$, $X_{uu} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$, $X_{uv} = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}$, $X_{vv} = ($

1^{st} fundamental form: $g_{uu} = X_u \cdot X_u = 1 + 4(u-v)^2$

$g_{uv} = g_{vu} = X_u \cdot X_v = -1 - 2u(2u-2v)$

$g_{vv} = X_v \cdot X_v = 2 + 4u^2$

2^{nd} fundamental form: since $\nu = \dfrac{X_u \times X_v}{\| X_u \times X_v \|} = \dfrac{(2v-2u,\ 2v,\ 1)}{\sqrt{4(u-v)^2 + 4v^2 + 1}}$

$$h_{uu} = -\langle X_{uu}, \nu \rangle = -\frac{2}{\sqrt{4(u-v)^2 + 4v^2 + 1}}$$

$$h_{vu} = h_{uv} = -\langle X_{uv}, \nu \rangle = \frac{2}{\sqrt{4(u-v)^2 + 4v^2 + 1}}$$

$$h_{vv} = -\langle X_{vv}, \nu \rangle = 0$$

(b) Set $S_{\frac{\partial}{\partial u}} = H_{uu}\frac{\partial}{\partial u} + H_{uv}\frac{\partial}{\partial v}$ Then $H_{ij} = (g_{ij})^{-1} h_{ij}$

$S_{\frac{\partial}{\partial v}} = H_{vu}\frac{\partial}{\partial u} + H_{vv}\frac{\partial}{\partial v}$

$$= \frac{1}{(1+4(u-2v)^2)^{\frac{3}{2}}} \begin{pmatrix} -2+12u^2-8uv & 4+8u^2 \\ -8uv+8v^2 & 2+8u^2-8u \end{pmatrix}$$

$\det(g_{ij}) = 1 + 4(u-2v)^2$

(c) Gauss curvature $K = \dfrac{\det(h_{ij})}{\det(g_{ij})} = -\dfrac{4}{(1+4(u-2v)^2)^{3/2}}$

Mean curvature $H = \frac{1}{2} g^{ij} h_{ij} = \frac{1}{2} \operatorname{tr} H = \dfrac{20u^2 - 16uv}{2(1+4(u-2v)^2)^{3/2}}$

2. Let M be a regular surface in $\mathbb{R}^3$, $g_{11} = g_{22} = e^{\phi}$. $g_{12} = 0$. $\phi = \phi(u^1,$

(a) $\Gamma^1_{11} = \frac{1}{2} g^{11} \cdot g_{11,1} = \frac{1}{2}\partial_{u^1}(\log e^{\phi}) = \frac{1}{2}\phi_{u^1}$, $\Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{2} g^{11} \cdot g_{11,2} = \frac{1}{2}\phi_{u^2}$

$\Gamma^1_{22} = \frac{1}{2} g^{11} \cdot (-g_{22,1}) = -\frac{1}{2}\phi_{u^1}$, $\Gamma^2_{11} = \frac{1}{2} g^{22} \cdot (-g_{11,2}) = -\frac{1}{2}\phi_{u^2}$

$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2} g^{22} \cdot g_{22,1} = \frac{1}{2}\phi_{u^1}$. $\Gamma^2_{22} = \frac{1}{2} g^{22}(-g_{22,2}) = -\frac{1}{2}\phi_{u^2}$

(b) $\ddot{\gamma}^i(t) + \Gamma^i_{jk}(\gamma(t))\, \dot{\gamma}^j(t)\, \dot{\gamma}^k(t) = 0.$

Let $\gamma(t) = (t+a,\ 2t+b)$, set $i=j=k=1$, then

$$\frac{1}{2}\phi_{u_1}\cdot 1 = 0 \Rightarrow \phi_{u_1} = 0$$

set $i=j=k=2$, then

$$-\frac{1}{2}\phi_{u_2}\cdot 4 = 0 \Rightarrow \phi_{u_2} = 0.$$

Thus ϕ is a constant if $\gamma(t)$ is a geodesic.

(c). If $U = \dfrac{X_1 \times X_2}{|X_1 \times X_2|}$, then

$$h_{ij} = \begin{pmatrix} X_{11}\cdot U & X_{12}\cdot U \\ X_{21}\cdot U & X_{22}\cdot U \end{pmatrix}. \quad \text{Since } (g_{ij})^{-1} = \begin{pmatrix} e^{-\phi} & 0 \\ 0 & e^{-\phi} \end{pmatrix}$$

$$H = \frac{1}{2}\operatorname{tr}\left((g_{ij})^{-1} h_{ij}\right) = \frac{1}{2}\operatorname{tr}\begin{pmatrix} e^{-\phi}X_{11}\cdot U & e^{-\phi}X_{12}\cdot U \\ e^{-\phi}X_{21}\cdot U & e^{-\phi}X_{22}\cdot U \end{pmatrix} = \frac{1}{2}e^{-\phi}(X_{11}+X_{22})\cdot U$$

Then $\quad 2e^{\phi}HU = [(X_{11}+X_{22})\cdot U]\,U$

Since $(X_{11}+X_{22})\cdot X_1 = X_{11}\cdot \cancel{X_1} + X_1\cdot X_{22} = X_{21}\cdot X_2 + X_1\cdot X_{22} = (X_1\cdot X_2)_{,2} = 0.$

similarly $(X_{11}+X_{22})\cdot X_2 = 0.$

Thus $2e^{\phi}HU = X_{11} + X_{22}.$

3. (a) By the variation of area preserving the same volume, we know that the surface with constant mean curvature will active the least area. (If Gauss curvature is positive).

And the compact embedded CMC surfaces are just the standard spheres.

Since $K \geq \frac{1}{9}$, $H \geq \frac{1}{3}$. The bigger the gauss curvature, the smaller the area for the standard sphere.

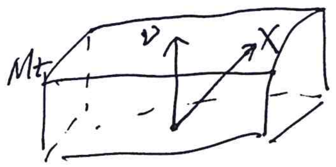
When $H = \frac{1}{3}$, $r = 3$, $\text{area}(M) = V = 4\pi\cdot 3^2$. $\quad 4\pi r^2 > \frac{4\pi}{3}r^3$ for r

(We ~~have also~~ used Alexandrov's thm: if M is a cpt embedded surface of constant mean curvature, then M is a standard sphere.)

3. The 1^{st} ~~variantional~~ variational formula:

$$A'(0) = -2\iint_M fH\,dA.$$

By divergence theorem, $\left.\frac{d}{dt}\right|_{t=0} Vol(\Omega_t) = \iint_{\partial\Omega = M} \langle X, \nu \rangle\, dA$ $\quad X$: ~~position~~ variation vec



$$= \iint_M \langle f\nu, \nu \rangle\, dA$$

If all Ω_t have the same volume, then $\iint_M f\,dA = 0$

$$\iint (H-c)^2 dA = \iint_M (H^2 - 2cH + c^2)\, dA = 0 \Rightarrow H = C = \text{const}.$$

where $c = \frac{1}{A}\iint_M H\,dA.$

$$\left(\begin{array}{l} \text{Since } \iint f\,dA = \iint (H-c)\,dA = 0. \\ \Rightarrow \iint f(H-c)\,dA = 0 \Rightarrow \iint (H-c)^2 = 0 \Rightarrow H = C \end{array} \right).$$

4. (a) $\chi(M) = \chi\big((M \setminus (S_1 \cup S_2)) \cup (S_1 \cup S_2)\big).$

$$= \chi(M \setminus (S_1 \cup S_2)) + \chi(S_1 \cup S_2)$$

By Gauss-Bonnet, $\int_{S_1 \cup S_2} K\,dS = 4\pi = 2\pi \times 2$

Thus $\chi(S_1 \cup S_2) = 2$. Then $\chi(M) = \chi(M \setminus (S_1 \cup S_2)) + 2$.

(b). Since N intersect P_1, P_2 orthogonally,

thus the ~~normal~~ geodesic curvature of α_i in N is zero. $(i = 1, 2)$.

(The curvature direction of α_i coincide with normal direction.

Then $\int_{N_0} K\,dS = 2\pi\chi(N_0) = 2\pi(\chi(M) - 2)$.

Since N_0 is homeomorphic to $M \setminus (S_1 \cup S_2)$.